Int. J. Heat Mass Transfer Vol. 15, pp. 351–354. Pergamon Press 1972. Printed in Great Britain

SHORTER COMMUNICATIONS

EXTERNAL NATURAL CONVECTION ABOUT TWO-DIMENSIONAL BODIES WITH CONSTANT HEAT FLUX

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(*Received* 14 *January* 1971 *and in revised form* 14 *May* 1971)

NOMENCLATURE

f_1, universal non-dimensional stream function;
F, transformed non-dimensional stream function;
$F_j(\eta)$, series coefficient non-dimensional stream functions;
g, acceleration due to gravity;
Gr^*, modified Grashof number;
Gr, Grashof number;
h_1, universal non-dimensional temperature;
H, transformed non-dimensional temperature;
$H_j(\bar{\eta})$, series coefficient non-dimensional stream functions;
k, thermal conductivity;
$K(\bar{\xi})$, principal function;
K_r, body parameters;
L, characteristic length;
$\overline{Nu}$, average Nusselt number;
q, heat flux constant;
$S(x)$, sine of the angle between the body force vector and a normal to the surface of the body;
T, temperature;
T_o, temperature of surrounding medium;
T_w, temperature at the boundary surface;
u, velocity component associated with increasing x;
$\bar{u}$, non-dimensional velocity component associated with increasing ξ;
v, velocity component associated with increasing y;
$\bar{v}$, non-dimensional velocity component associated with increasing η;
x, distance along the body measured from the lower stagnation point;
y, distance normal to the body.

Greek letters

α, exponent associated with body shape;
β, coefficient of thermal expansion;
ψ, stream function;
$\bar{\psi}$, non-dimensional stream function;
κ, coefficient of thermal conductivity;
ν, kinematic viscosity;
σ, Prandtl number, ν/κ;
θ, angle between body force vector and a normal to the body;
Θ, non-dimensional temperature;
ξ, non-dimensional distance along the body;
$\bar{\xi}$, transformed non-dimensional distance along the body;
η, non-dimensional distance normal to the body;
$\bar{\eta}$, transformed non-dimensional distance normal to the body.

INTRODUCTION

In 1957 Gortler [1] introduced transformations of the steady, two-dimensional, isothermal, incompressible boundary layer equations designed to yield rapidly convergent series solutions applicable to an extensive class of body contours. Recently Saville and Churchill [2] introduced an analysis closely patterned on Gortler's which described laminar free convection near isothermal horizontal cylinders with fairly arbitrary body contours. The work that follows outlines the transformations and choice of independent variable appropriate to this class of body contours when the heat flux at the bounding surface is maintained as constant. The solutions of the two, coupled, partial differential equations for the temperature and stream function are then represented by series which are universal with respect to the body contours within a specified class of body shapes (e.g. round nosed cylinders). In particular the solution for the boundary layer flow in the vicinity of a horizontal circular cylinder, under the prescribed conditions, is discussed.

THE EQUATIONS

With the usual free convection assumptions the boundary-layer equations of momentum, energy and continuity take the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_0)\sin\theta + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2} \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial y}{\partial y} \tag{3}$$

to be examined under boundary conditions:

$$u=v=0:\quad -k(\partial T/\partial y)=q=\text{constant on } y=0$$
$$u\to 0;\quad T\to T_0 \text{ as } y\to\infty. \tag{4}$$

The flow is characterized by the modified Grashof number $Gr^*=(g\beta qL^4/\nu^2 k)$ and appropriate non-dimensionalizing transformations are

$$\bar{u}=\frac{Lu}{\nu Gr^{*\frac{2}{5}}};\ \bar{v}=\frac{Lv}{\nu Gr^{*\frac{1}{5}}}(\psi=\nu Gr^{*\frac{1}{5}}\psi)$$

$$T-T_0=-\frac{q}{k}Gr^{*-\frac{1}{5}}L\theta;\quad \xi=\frac{x}{L};\quad \eta=\frac{yGr^{*\frac{1}{5}}}{L} \tag{5}$$

leading to

$$\bar{u}\frac{\partial\bar{u}}{\partial\xi}+\bar{v}\frac{\partial\bar{u}}{\partial\eta}=-\theta S(\xi)+\frac{\partial^2\bar{u}}{\partial\eta^2} \tag{6}$$

$$\bar{u}\frac{\partial\theta}{\partial\xi}+\bar{v}\frac{\partial\theta}{\partial\eta}=\frac{1}{\sigma}\frac{\partial^2\theta}{\partial\eta^2} \tag{7}$$

$$\left.\begin{aligned}&\bar{u}=\bar{v}=0;\quad \frac{\partial\theta}{\partial\eta}=1 \text{ on } \eta=0\\ &\bar{u}\to 0;\quad \theta\to 0 \text{ as } \eta\to\infty\end{aligned}\right\} \tag{8}$$

The further transformations which carry the inherent advantages discussed by Gortler [1] and obtained by similar argument are

$$\bar{\xi}=\int_0^{\xi}[S(t)]^{\frac{1}{4}}\mathrm{d}t;\ \bar{\eta}=\frac{\eta S^{\frac{1}{5}}}{(5\bar{\xi})^{\frac{1}{5}}}$$

$$\bar{\psi}=(5\bar{\xi})^{\frac{4}{5}}F(\bar{\xi},\bar{\eta});\quad \theta=\frac{(5\bar{\xi})^{\frac{1}{5}}}{S^{\frac{1}{5}}}H(\bar{\xi},\bar{\eta}) \tag{9}$$

which yield

$$F_{\bar{\eta}\bar{\eta}\bar{\eta}}+4FF_{\bar{\eta}\bar{\eta}}-3F_{\bar{\eta}}^{\,2}\,.\,K(\bar{\xi})-H=5\bar{\xi}[F_{\bar{\eta}}F_{\bar{\eta}}-F_{\bar{\eta}\bar{\eta}}F_{\bar{\xi}}] \tag{10}$$

$$\frac{1}{\sigma}H_{\bar{\eta}\bar{\eta}}+4FH_{\bar{\eta}}-F_{\bar{\eta}}H=5\bar{\xi}[F_{\bar{\eta}}H_{\bar{\xi}}-F_{\bar{\xi}}H_{\bar{\eta}}] \tag{11}$$

where

$$K(\bar{\xi})=1+\tfrac{1}{12}\frac{\bar{\xi}}{S}\frac{\mathrm{d}S}{\mathrm{d}\bar{\xi}} \tag{12}$$

with boundary conditions

$$H_{\bar{\eta}}(\bar{\xi},0)=1:\quad F(\bar{\xi},0)=F_{\bar{\eta}}(\bar{\xi},0)=F_{\bar{\eta}}(\bar{\xi},\infty)$$
$$=H(\bar{\xi},\infty)=0. \tag{13}$$

The function $K(\bar{\xi})$ is directly associated with the shape of the body contour and will be termed the principal function. Moreover for cases other than $K(\bar{\xi})=$ constant (i.e. cases allowing similarity solutions) the principal function can be expanded in series form

$$K(\bar{\xi})=\sum_{j=0}^{\infty}K_j\bar{\xi}^{\alpha j} \tag{14}$$

where the numbers K_0 and α depend only on the class of the body shape, not on the details of the contour.

SERIES SOLUTION

The solution now requires the expansion of F and H in a similar form to that of the principal function i.e.

$$F(\bar{\xi},\bar{\eta})=\sum_{j=0}^{\infty}F_j(\bar{\eta})\bar{\xi}^{\alpha j};\quad H(\bar{\xi},\bar{\eta})=\sum_{j=0}^{\infty}H_j(\bar{\eta})\bar{\xi}^{\alpha j}. \tag{15}$$

The following system of equations result

$$\left.\begin{aligned}&0(1)A\ F_0'''+4F_0F_0''-3K_0F_0'^2-H_0=0\\ &\frac{1}{\sigma}H_0''+4F_0H_0'-F_0'H_0=0\end{aligned}\right\} \tag{16}$$

$$0(\bar{\xi}^{\alpha j})\ F_j'''+4F_0F_j''-F_0'F_j'(3\,.\,2\,.\,K_0+5\alpha j)$$
$$+F_0''F_j(4+5\alpha j)-3K_jF_0'^2-H_j=RF_j \tag{17}$$

where $RF_j=\sum_{k=1}^{j-1}\{(5\alpha k+3K_0)F_k'F_{j-k}'-(4+5\alpha k)F_kF''_{j-k}$

$$+3K_{j-k}\sum_{i=0}^{k}F_i'F_{k-i}'\}$$

$$\frac{1}{\sigma}H_j''+4F_0H_j'-F_0'H_j(1+5\alpha j)+F_jH_0'(4+5\alpha j)$$
$$-F_j'H_0=\sum_{k=1}^{j-1}\{(1+5\alpha k)H_kF'_{j-k}-(4+5\alpha k)F_kH'_{j-k}\} \tag{18}$$

with the appropriate boundary conditions

$$\begin{aligned}&0(1)\quad H_0'(0)=1;\quad F_0(0)=F_0'(0)=H_0(\infty)=F_0'(\infty)=0\\ &0(\bar{\xi}^{\alpha j})\quad H_j'(0)=F_j(0)=F_j'(0)=H_j(\infty)=F_j'(\infty)=0.\end{aligned} \tag{19}$$

The form of equations (17) and (18) allow us to scale out of the problem the coefficients of the principal function $K_j(j\geqslant 1)$ and develop a set of functions universal with respect to a particular class of body shapes and a specific Prandtl number. The solution for any body contour within a given class can then be obtained by assembling a series of the universal functions with coefficients particular to the body contour under discussion.

APPLICATION

For sharp-nosed cylinders $K_0=1$, $\alpha=1$ so that the solution of Sparrow and Gregg [3] should provide a good first approximation for all such bodies so long as the dependent and independent variables are interpreted appropriately.

There remains the class of blunt nosed bodies, for which $K_0=\frac{4}{3}$, $\alpha=\frac{8}{5}$. The horizontal circular cylinder is a particular example and in this case

$$K(\bar{\xi}) = K_0 + K_1\bar{\xi}^{\frac{8}{5}} + \ldots = \tfrac{4}{3} - (\tfrac{4}{39})(\tfrac{5}{4}\bar{\xi})^{\frac{8}{5}}\ldots \quad (20)$$

Setting

$$\begin{aligned} F(\bar{\xi},\bar{\eta}) &= F_0(\bar{\eta}) + K_1 f_1(\bar{\eta})\bar{\xi}^{\frac{8}{5}} + \ldots \\ H(\bar{\xi},\bar{\eta}) &= H_0(\bar{\eta}) + K_1 h_1(\bar{\eta})\bar{\xi}^{\frac{8}{5}} + \ldots \end{aligned} \quad (21)$$

yields for the universal functions F_0, H_0, f_1, h_1

$$\left.\begin{aligned} &F_0''' + 4F_0F_0'' - 4F_0'^2 - H_0 = 0 \\ &\frac{1}{\sigma}H_0'' + 4F_0H_0' - F_0'H_0 \quad = 0 \end{aligned}\right\} \quad (22)$$

with boundary conditions

$$H_0'(0) = 1;\quad F_0(0) = F_0'(0) = H_0(\infty) = F_0'(\infty) = 0 \quad (23)$$

and

$$\left.\begin{aligned} &f_1''' + 4F_0f_1'' - 16F_0'f_1' + 12F_0''f_1 - 3F_0'^2 - h_1 = 0 \\ &\frac{1}{\sigma}h_1'' + 4F_0h_1' + 12f_1H_0' - 9F_0'h_1 - f_1'H_0 \quad = 0 \end{aligned}\right\} \quad (24)$$

with boundary conditions

$$f_1(0) = f_1'(0) = f_1'(\infty) = h_1'(0) = h_1(\infty) = 0. \quad (25)$$

The O(1) equations for F_0, H_0 are essentially the solution for laminar free convection in the vicinity of a constant heat flux lower stagnation point. Again it is the interpretation of the dependent and independent variables which enables us to consider the solutions for F_0, H_0 as valid first approximations over that part of the body for which boundary layer assumptions are appropriate. We would not expect relevant information from this solution much beyond 150° from the lower stagnation point in the case of the circular cylinder.

RESULTS

Numerical solutions of the system of equations (22)–(25) have been established and details are available from the author. The appropriate initial values for a range of Prandtl number are listed in Table 1.

Table 1.

Pr	$F_0''(0)$	$-H_0(0)$	$-f_1''(0)$	$-h_1(0)$
0·1	1·5654	2·7806	0·11769	0·09198
0·7	0·8084	1·5206	0·03575	0·03627
1·0	0·7117	1·3746	0·02758	0·02944
10	0·3047	0·7717	0·00446	0·00662
100	0·1258	0·4671	0·00058	0·00127

To demonstrate the effectiveness of the choice of variables in producing a series solution, of which the first term closely approximates the solution, velocity and temperature profiles at 90° from the lower stagnation point have been calculated from one-term and two-term approximations. The results are plotted in Figs. 1 and 2.

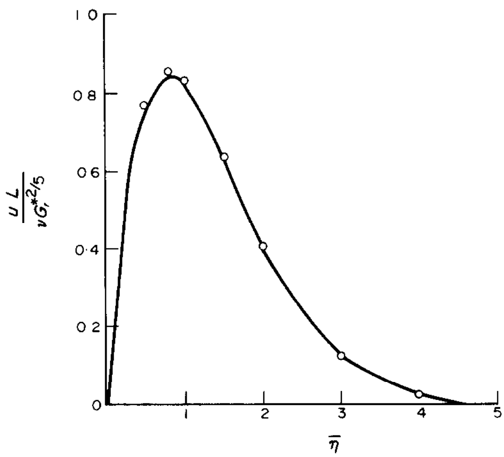

FIG. 1. Dimensionless velocity profile at 90° from lower stagnation point for $Pr = 0{\cdot}7$.
— one-term approximation
○ two-term approximation

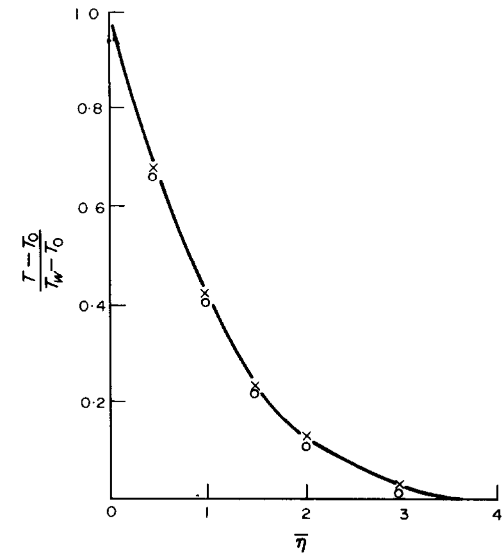

FIG. 2. Temperature profile at 90° from lower stagnation point for $Pr = 0{\cdot}7$.
— one-term approximation
○ two-term approximation

The average Nusselt number $\overline{Nu}$ (averaged over the perimeter of the cylinder) is given by

$$\overline{Nu} = -\frac{(5\bar{\xi})^{\frac{4}{5}}}{4\pi H(0)}Gr^{*\frac{1}{5}} = -\frac{(5\bar{\xi})^{\frac{4}{5}}}{4\pi H(0)}\overline{Gr}^{\frac{1}{4}}\,\overline{Nu}^{\frac{1}{4}}$$

where $\overline{Gr}$ is the Grashof number based on the typical temperature difference employed in defining $\overline{Nu}$. We have then

$$\overline{Nu}\,\overline{Gr}^{-\frac{1}{4}} = -\frac{(5\bar{\xi})}{[4\pi H(0)]^{\frac{1}{4}}}.$$

Since for the circular cylinder

$$\bar{\xi}(\pi) = \int_0^{\pi} (\sin z)^{\frac{1}{4}}\,dz = (\sqrt{\pi})\frac{\Gamma(5/8)}{\Gamma(9/8)} \simeq 2{\cdot}70$$

$\overline{Nu}\,\overline{Gr}^{-\frac{1}{4}}$ can be evaluated and in fact yields the value 0·338 when based on a one-term evaluation of $H(0)$ as opposed to 0·345 when based on a two-term evaluation of $H(0)$. A comparison of the ratio of this quantity to the equivalent quantity obtained by Saville and Churchill [2] for the isothermal case yields a one-term ratio value of 1·13 and a two-term value of 1·14. This result correlates almost exactly with the vertical plate results of Sparrow and Gregg [3].

Finally, representations of one-term velocity and temperature profiles for the appropriate Prandtl numbers are presented in Figs. 3 and 4.

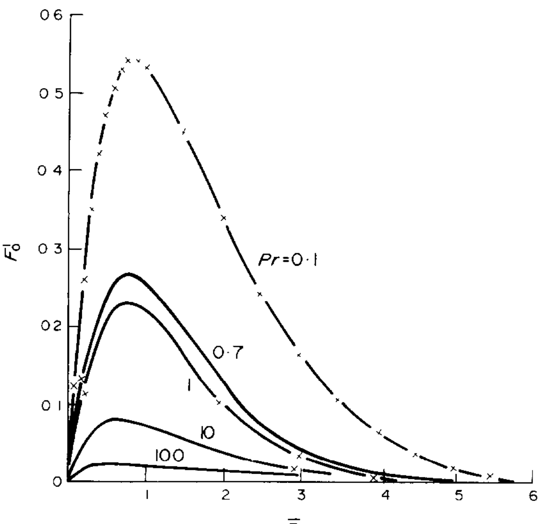

FIG. 3. F_0' profiles for $Pr = 0{\cdot}1$, 0·7, 1, 10 and 100.

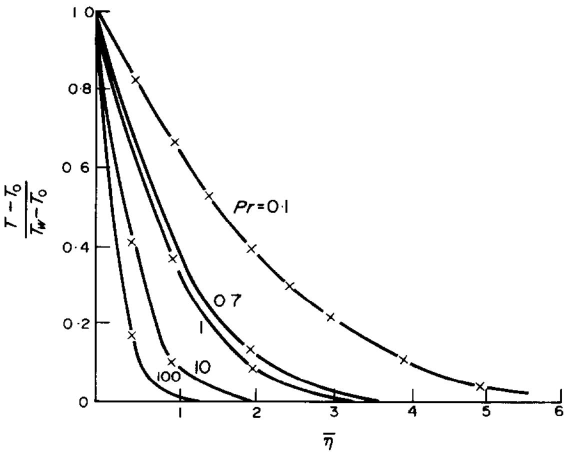

FIG. 4. Temperature profiles for $Pr = 0{\cdot}1$, 0·7, 1, 10 and 100.

REMARKS

It is to be noted that the differential equations and boundary conditions for the zero order terms in the Gortler representation are the same as for the Blasius in representation. The difference, a crucial one, at this level lies solely in the interpretation of the independent and dependent variables, which have been established with a view to concentrating as much information as possible into the zero-order solution. It is this interpretation which leads us to anticipate the rapidity of convergence of the series solution. The magnitude of first-order correction terms in the case of the horizontal circular cylinder has been shown to support this conclusion. In fact, it may well be that for practical purposes the zero-order term is sufficiently accurate.

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